

# Weighted cumulative entropies: An extension of CRE and CE

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**Abstract.** We generalize the weighted cumulative entropies (WCRE and WCE), introduced in [5], for a system or component lifetime. Representing properties of cumulative entropies, several bounds and inequalities for the WCRE is proposed.

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## 1 Introduction. The weighted cumulative entropies

An important measure of the uncertainty is entropy, commonly termed the Shannon information measure, [10]. The current paper deals with *weighted* entropy; for the definition and initial results on weighted entropy the reader is referred to [1, 4].

Furthermore, the entropy of the residual lifetime  $X_t = [X - t | X > t]$  as a dynamic measure of uncertainty was considered in [3]. Recently, further progress as introducing cumulative entropy, cumulative residual entropy and weighted cumulative entropy was made in [7, 2, 5].

The purpose of this work is to obtain a number of results for weighted cumulative entropies in the case where the weight is a general non-negative function.

Let  $X$  be a non-negative absolutely continuous random variable describing a component failure time, with the probability density function (PDF),  $f(x)$ , the cumulative distribution function (CDF),  $F(x) = P(X \leq x)$ , and the survival function (SF),  $\bar{F}(x) = P(X > x)$ .

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**Definition 1.1** Given a function  $x \in \mathbb{R} \mapsto \phi(x) \geq 0$ , and an RV  $X : \Omega \rightarrow \mathbb{R}$ , with a PDF  $f$ , the **weighted cumulative residual entropy** (WCRE) of  $X$  (or  $F$ ) with weight function (WF)  $\phi$  is defined by

$$\mathcal{E}_\phi^w(X) = \mathcal{E}_\phi^w(F) = - \int_{\mathbb{R}_+} \phi(x) \mathbb{P}(|X| > x) \log \mathbb{P}(|X| > x) dx. \quad (1.1)$$

Note that a standard agreement  $0 = 0 \cdot \log 0 = 0 \cdot \log \infty$  is adopted throughout the paper.

Given the CDF,  $\mathbf{x} \in \mathbb{R}_+ \mapsto F(\mathbf{x}) \in [0, 1]$ , with WF  $\phi$ , the **weighted cumulative entropy** (WCE) of non-negative random lifetime  $X$  is defined by

$$\bar{\mathcal{E}}_\phi^w(X) = \bar{\mathcal{E}}_\phi^w(F) = - \int_{\mathbb{R}_+} \phi(x) \mathbb{P}(|X| \leq x) \log \mathbb{P}(|X| \leq x) dx. \quad (1.2)$$

Particularly when  $\phi(x) = x$  the WCRE and WCE in (1.1) and (1.2) can be turned out as (8) and (9) in [5]. In what follows, we intend to use the same abbreviation as in [5] for the weighted cumulative residual and weighted cumulative entropies.

**Example 1.1** (WCRE of the uniform distribution) Consider an RV  $X$  with uniform distribution in the interval  $a < b$ ,  $f(x) = \frac{1}{(b-a)}$ . Then the WCRE is the following:

$$\begin{aligned} \mathcal{E}_\phi^w(F) &= - \int_a^b \phi(x) \left(1 - \frac{x}{b-a}\right) \log \left(1 - \frac{x}{b-a}\right) dx \\ &= (b-a) \int_{\frac{b-2a}{b-a}}^{\frac{-a}{b-a}} \phi((b-a)(1-y)) y \log y dy. \end{aligned}$$

In particular, with  $\phi(x) = x$ , one obtains:

$$\begin{aligned} \mathcal{E}_\phi^w(F) &= \frac{1}{36(a-b)} \left[ 24a^3 \log \left( \frac{2a-b}{a-b} \right) - 15a^2b - a^3 - 5b^3 + 6 \log \left( \frac{2a-b}{a-b} \right) b^3 \right. \\ &\quad \left. - 18ab^2 \log \left( \frac{2a-b}{a-b} \right) + 21ab^2 + 6a^3 \log \left( \frac{a}{a-b} \right) - 18a^2b \log \left( \frac{a}{a-b} \right) \right]. \end{aligned}$$

**Example 1.2** (WCRE of the Gaussian distribution) Let  $g(x)$  be the Gaussian PDF with mean  $\mu$  and variance  $\sigma^2$ . Therefore, the SF is obtained as  $\bar{G}(x) = \text{erfc}(\frac{x-\mu}{\sigma})$ . Here  $\text{erfc}(x)$  is the complementary error function:

$$\text{erfc}(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp(-\frac{t^2}{2}) dt.$$

In accordance with (1.1) we obtain:

$$\mathcal{E}_\phi^w(G) = - \int_0^\infty \phi(x) \text{erfc}\left(\frac{x-\mu}{\sigma}\right) \log \text{erfc}\left(\frac{x-\mu}{\sigma}\right) dx.$$

Given an RV  $X$  with CDF  $F(x)$  and SF  $\bar{F}(x)$ , set,

$$m_F^w(t) = \frac{1}{\bar{F}(t)} \int_t^\infty \phi(x) \bar{F}(x) dx \quad \text{and} \quad \bar{m}_F^w(t) = \frac{1}{F(t)} \int_0^t \phi(x) F(x) dx.$$

Pictorially  $m_F^w(t)$  represents the weighted mean inactivity time (WMIT) and then  $\bar{m}_F^w(t)$  the weighted mean residual time (WMRT).

**Lemma 1.1** (Cf. Proposition 2.1 from [5].) *Let  $X$  be an absolutely continuous RV. Then*

$$\mathcal{E}_\phi^w(F) = \mathbb{E}(m_F^w(X)), \quad (1.3)$$

and

$$\bar{\mathcal{E}}_\phi^w(F) = \mathbb{E}(\bar{m}_F^w(X)). \quad (1.4)$$

**Proof.** The proof follows directly with the same methodology in [5] but replacing  $\phi(x)$  in  $x$ .

**Definition 1.2** *Given two functions  $x \in \mathbb{R}^+ \mapsto \bar{F}(x) \in [0, 1]$  and  $x \in \mathbb{R}^+ \mapsto \bar{G}(x) \in [0, 1]$ , the relative WCRE of  $\bar{G}$  relative to  $\bar{F}$  for given WF  $\phi$  is defined by*

$$D_\phi^w(\bar{F} \parallel \bar{G}) = \int_{\mathbb{R}_+} \phi(x) \bar{F}(x) \log \frac{\bar{F}(x)}{\bar{G}(x)} dx. \quad (1.5)$$

Alternatively,  $D_\phi^w(\bar{F} \parallel \bar{G})$  can be termed as weighted Kullback Leibler divergence between SFs  $\bar{F}$ ,  $\bar{G}$  with WF  $\phi$ .

Paying homage to the Theorem 1.1 in [11], with similar methodology the following assertion holds true. We omit the proof.

**Theorem 1.1** *Given SFs  $\bar{F}$  and  $\bar{G}$  in  $[0, 1]$ , assume that a WF  $x \in \mathbb{R}_+ \mapsto \phi(x) \geq 0$  obeys*

$$\int_{\mathbb{R}_+} \phi(x) [\bar{F}(x) - \bar{G}(x)] dx \geq 0.$$

Then

$$D_\phi^w(\bar{F} \parallel \bar{G}) \geq 0. \quad (1.6)$$

The equality occurs iff  $[\frac{\bar{G}}{\bar{F}} - 1]\phi = 0$  for  $\bar{F}$ -almost all  $x \in \mathbb{R}_+$ .

**Theorem 1.2** (Estimating the WCRE via a uniform distribution, cf. Theorem 1.2 in [11].) *Assume that RV  $X$  takes at most  $m$  values and set  $p_i = \mathbb{P}(X = i)$  and  $\bar{p}_i = \sum_{j=1}^i p_j$ . Suppose that for given  $0 < \beta \leq 1$*

$$\sum_{i=1}^m \phi(i) [\bar{p}_i - \beta i] \geq 0,$$

*Then*

$$-\sum_{i=1}^m \phi(i) \bar{p}_i \log \bar{p}_i \leq -\log \beta \sum_{i=1}^m \phi(i) \bar{p}_i - \sum_{i=1}^m \phi(i) \bar{p}_i \log i,$$

*with equality iff for all  $i = 1 \dots m$ ,  $\phi(i) [\bar{p}_i - \beta i] = 0$ .*

*Furthermore, assume for given  $\alpha, \beta \in \mathbb{R}$ :*

$$\int_{\mathbb{R}_+} \phi(x) [\bar{F}(x) - (\alpha - \beta x)] dx \geq 0,$$

*The following assertion for non-negative RV  $X$  holds true:*

$$\mathcal{E}_\phi^w(X) \leq - \int_{\mathbb{R}_+} \phi(x) \bar{F}(x) \log(\alpha - \beta x) dx.$$

*Here the equality holds iff  $\phi(x) [\bar{F}(x) - (\alpha - \beta x)] = 0$ .*

**Definition 1.3** *Consider a random vector  $\mathbf{X} = (X_1, \dots, X_n) : \Omega \rightarrow \mathbb{R}^n$  with join survival function  $\bar{F}(\mathbf{x}) = \mathbb{P}[X_1 > x_1, \dots, X_n > x_n]$ . The WCRE and WCE for given WF  $\phi$ , are defined by*

$$\begin{aligned} \mathcal{E}_\phi^w(\mathbf{X}) &= - \int_{\mathbb{R}_+^n} \phi(\mathbf{x}) P[|\mathbf{X}| > \mathbf{x}] \log P[|\mathbf{X}| > \mathbf{x}] d\mathbf{x}, \\ \bar{\mathcal{E}}_\phi^w(\mathbf{X}) &= - \int_{\mathbb{R}_+^n} \phi(\mathbf{x}) P[|\mathbf{X}| \leq \mathbf{x}] \log P[|\mathbf{X}| \leq \mathbf{x}] d\mathbf{x}. \end{aligned} \tag{1.7}$$

*Here  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbb{R}_+^n = (x_i \in \mathbb{R}^n, x_i \geq 0)$ .*

*Let  $(X_1, X_2) \in \mathbb{R}^2 \mapsto \phi(x_1, x_2)$  be a given bivariate WF. The **conditional** WCRE of  $X_1$  given  $X_2$  is defined by*

$$\begin{aligned} \mathcal{E}_\phi^w(X_1|X_2) &= - \int_{\mathbb{R}_+^2} \phi(x_1, x_2) \mathbb{P}(|X_1| > x_1, |X_2| > x_2) \log \frac{\mathbb{P}(|X_1| > x_1, |X_2| > x_2)}{\mathbb{P}(|X_2| > x_2)} dx_1 dx_2. \end{aligned} \tag{1.8}$$

and the **mutual** WCRE between non-negative random vectors  $\mathbf{X}$  with joint SF  $\bar{F}$  and marginal  $\bar{F}_i$ ,  $i = 1, \dots, n$  by

$$\begin{aligned}\tau_\phi^w(\mathbf{X}) &:= D_\phi^w(\bar{F} \| \bar{F}_1 \otimes \dots \otimes \bar{F}_n) \\ &= \int_{\mathbb{R}_+^n} \phi(\mathbf{x}) \bar{F}(\mathbf{x}) \log \frac{\bar{F}(\mathbf{x})}{\bar{F}_1(x_1) \dots \bar{F}_n(x_n)} d\mathbf{x}.\end{aligned}\tag{1.9}$$

**Lemma 1.2** (Bounding on conditional WCRE, Cf. Lemma 1.1 in [11].) *Let  $\mathbf{X}_1^2 = (X_1, X_2)$  be a pair of RVs with the joint SF  $\bar{F}(x_1, x_2)$  and marginal SFs  $\bar{F}_1(x_1)$ ,  $\bar{F}_2(x_2)$ . Suppose that the WF  $\phi$  obeys*

$$\int_{\mathbb{R}_+^2} \phi(\mathbf{x}_1^2) \bar{F}(\mathbf{x}_1^2) [\bar{F}(x_1|x_2) - 1] d\mathbf{x}_1^2 \leq 0\tag{1.10}$$

Then

$$\mathcal{E}_\phi^w(\mathbf{X}_1^2) \leq \mathcal{E}_{\psi_2}^w(X_2) \text{ or, equivalently, } \mathcal{E}_\phi^w(X_1|X_2) \geq 0.\tag{1.11}$$

Here  $\psi_2 = \int_{\mathbb{R}_+} \phi(\mathbf{x}_1^2) \bar{F}(x_1|x_2) dx_1$  and the equality holds true iff  $\phi(\mathbf{x}_1^2) [\bar{F}(x_1|x_2) - 1] = 0$  for all  $\mathbf{x} \in \mathbb{R}_+^2$ .

Furthermore, consider triple RVs  $\mathbf{X}_1^3 = (X_1, X_2, X_3)$  and assume that

$$\int_{\mathbb{R}_+^3} \phi(\mathbf{x}_1^3) \bar{F}(\mathbf{x}_1^3) [\bar{F}(x_1|\mathbf{x}_2^3) - 1] d\mathbf{x}_1^3.\tag{1.12}$$

Then

$$\mathcal{E}_\phi^w(\mathbf{X}_1^3) \leq \mathcal{E}_{\psi_{23}}^w(\mathbf{X}_2^3) \text{ or, equivalently, } \mathcal{E}_\phi^w(X_1|\mathbf{X}_2^3) \geq 0.\tag{1.13}$$

Here  $\psi_{23} = \int_{\mathbb{R}_+} \phi(\mathbf{x}_1^3) \bar{F}(x_1|\mathbf{x}_2^3) dx_1$ . In (1.13) the equality holds true iff  $\phi(\mathbf{x}_1^3) [\bar{F}(x_1|\mathbf{x}_2^3) - 1] = 0$  for all  $\mathbf{x}_1^3 \in \mathbb{R}_+^3$ .

By subscribing  $\bar{F}$  in  $f$  in Theorem 1.3 in [11] with the same arguments, the following assertion, omitting the proof, is achieved.

**Theorem 1.3** (Sub-additivity of the WCRE, Cf. Theorem 1.3 in [11].) *Let  $\mathbf{X}_1^2 = (X_1, X_2)$  be a pair of RVs with join SF  $\bar{F}(\mathbf{x}_1^2)$  and marginal survival function  $\bar{F}_1(x_1)$ ,  $\bar{F}_2(x_2)$ . Moreover suppose that the WF  $(x_1, x_2) \in \mathbb{R}_+^2 \mapsto \phi(x_1, x_2) \geq 0$  obeys*

$$\int_{\mathbb{R}_+^2} \phi(\mathbf{x}_1^2) [\bar{F}(\mathbf{x}_1^2) - \bar{F}_1(x_1) \bar{F}_2(x_2)] d\mathbf{x}_1^2 \geq 0.\tag{1.14}$$

Then

$$\begin{aligned} \mathcal{E}_\phi^w(\mathbf{X}_1^2) &\leq \mathcal{E}_{\psi_1}^w(X_1) + \mathcal{E}_{\psi_2}^w(X_2), \text{ or, equivalently, } \mathcal{E}_\phi^w(X_1|X_2) \leq \mathcal{E}_{\psi_1}^w(X_1), \\ &\text{or, equivalently, } \tau_\phi^w(X_1 : X_2) \geq 0. \end{aligned} \quad (1.15)$$

The equality occurs iff  $X_1, X_2$  are independent modulo  $\phi$  i.e.  $\phi(\mathbf{x}_1^2) \left[ 1 - \frac{\bar{F}_1(x_1)\bar{F}_2(x_2)}{\bar{F}(\mathbf{x}_1^2)} \right] = 0$  for all  $\mathbf{x}_1^2 \in \mathbb{R}_+^2$ . Here  $\psi_1$  and  $\psi_2$  are emerging from conditional survival functions:

$$\psi_i = \int_{\mathbb{R}_+} \phi(\mathbf{x}_1^2) \bar{F}(x_j|x_i) dx_j, \quad i, j = 1, 2, \quad i \neq j. \quad (1.16)$$

For given WF  $\mathbf{X}_1^3 \in \mathbb{R}_+^3 \mapsto \phi(\mathbf{x}_1^2) \geq 0$ , define

$$\begin{aligned} \psi_{12}(\mathbf{x}_1^2) &= \int_{\mathbb{R}_+} \phi(\mathbf{x}_1^3) \bar{F}(x_3|\mathbf{x}_1^2) dx_3, \quad \mathbf{x}_1^2 \in \mathbb{R}_+^2. \\ \psi_1^{23}(x_1) &= \int_{\mathbb{R}_+^2} \phi(\mathbf{x}_1^3) \bar{F}(\mathbf{x}_2^3|x_1) d\mathbf{x}_2^3, \quad x_1 \in \mathbb{R}_+. \end{aligned} \quad (1.17)$$

and similarly define  $\psi_k^{ij}$  and  $\psi_{ij}$  for distinct labels  $1 \leq i, j, k \leq 3$ . Then as in [11], use  $\psi_{12}$  in (1.14) if the assumption

$$\int_{\mathbb{R}_+^3} \phi(\mathbf{x}_1^3) [\bar{F}(\mathbf{x}_1^2) - \bar{F}_1(x_1)\bar{F}_2(x_2)] \bar{F}(x_3|\mathbf{x}_1^2) d\mathbf{x}_1^2 \geq 0. \quad (1.18)$$

holds true, Then

$$\mathcal{E}_{\psi_{12}}^w(X_1|X_2) \leq \mathcal{E}_{\psi_1^{23}}^w(X_1). \quad (1.19)$$

Following the given assertions in [11], the analogue inequalities each requiring its own assumption are represented. Note that in the list of assumptions (1.15),(1.17),(1.22),(1.27) in [11] swap  $\bar{F}$  in  $f$ :

$$\begin{aligned} &\text{by Lemma 1.1, [11]: } 0 \leq \mathcal{E}_\phi^w(X_1|\mathbf{X}_2^3), \quad \text{assuming 1.17 (a modified form of 1.15),} \\ &\text{by Lemma 1.3, [11]: } \mathcal{E}_\phi^w(X_1|\mathbf{X}_2^3) \leq \mathcal{E}_{\psi_{12}}^w(X_1|X_2), \quad \text{assuming (1.27),} \\ &\text{by Theorem 1.3, [11]: } \mathcal{E}_{\psi_{12}}^w(X_1|X_2) \leq \mathcal{E}_{\psi_1^{23}}^w(X_1), \quad \text{assuming (1.22),} \\ &\text{by Lemma 1.2, [11]: } \mathcal{E}_{\psi_{12}}^w(X_1|X_2) \leq \mathcal{E}_\phi^w(\mathbf{X}_{1,3}|X_2), \quad \text{assuming (1.26),} \\ &\text{by Theorem 1.4, [11]: } \mathcal{E}_\phi^w(\mathbf{X}_{13}|X_2) \leq \mathcal{E}_{\psi_{12}}^w(X_1|X_2) + \mathcal{E}_{\psi_{23}}^w(X_3|X_2), \quad \text{assuming (1.27),} \\ &\text{by Theorem 1.5, [11]: } \mathcal{E}_\phi^w(\mathbf{X}_1^3) + \mathcal{E}_{\psi_2^{13}}^w(X_2) \leq \mathcal{E}_{\psi_{12}}^w(\mathbf{X}_1^2) + \mathcal{E}_{\psi_{23}}^w(\mathbf{X}_2^3), \quad \text{assuming (1.27).} \end{aligned} \quad (1.20)$$

Next we represent a number of results which are analogue assertions in [11], hence proofs omitted.

**Theorem 1.4** (Strong sub-additivity of the WCRE). *Given a triple of RVs  $\mathbf{X}_1^3 = (X_1, X_2, X_3)$ , assume that*

$$\int_{\mathbb{R}_+^3} \phi(\mathbf{x}_1^3) \left[ \bar{F}(\mathbf{x}_1^3) - \bar{F}(x_2) \prod_{i=1,3} \bar{F}(x_i|x_2) \right] d\mathbf{x}_1^3 \geq 0. \quad (1.21)$$

*is fulfilled. Then*

$$\mathcal{E}_\phi^w(\mathbf{X}_1^3) - \mathcal{E}_{\psi_{13}}^w(X_2) \leq \mathcal{E}_{\psi_{12}}^w(\mathbf{X}_1^2) + \mathcal{E}_{\psi_{23}}^w(\mathbf{X}_2^3). \quad (1.22)$$

*The equality holds iff RVs  $X_1$  and  $X_3$  are conditionally independent  $X_2$ .*

**Theorem 1.5** (a) (Convexity of relative WCRE). *Given a WF  $x \in \mathbb{R}_+ \mapsto \phi(x)$  and  $\lambda_1 \lambda_2 \in (0, 1)$  with  $\lambda_1 + \lambda_2 = 1$ , then*

$$\lambda_1 D_\phi^w(\bar{F}_1 \| \bar{G}_1) + \lambda_2 D_\phi^w(\bar{F}_2 \| \bar{G}_2) \geq D_\phi^w(\lambda_1 \bar{F}_1 + \lambda_2 \bar{F}_2 \| \lambda_1 \bar{G}_1 + \lambda_2 \bar{G}_2), \quad (1.23)$$

*with equality iff  $\lambda_1 \lambda_2 = 0$  or  $\bar{F}_1 = \bar{F}_2$  and  $\bar{G}_1 = \bar{G}_2$  modulo  $\phi$ .*

(b) (Data-processing inequality for relative WCRE). *Let  $(\bar{F}, \bar{G})$  be the pair of SFs and  $\phi$  a WF in  $\mathbb{R}_+$ . For given stochastic kernel  $\mathbf{\Pi} = (\Pi(x, y), x, y \in \mathbb{R}_+)$ , set  $\Psi(u) = \int_{\mathbb{R}_+} \phi(x) \Pi(u, x) dx$ . Then*

$$D_\Psi^w(\bar{F} \| \bar{G}) \geq D_\phi^w(\bar{F} \mathbf{\Pi} \| \bar{G} \mathbf{\Pi}) \quad (1.24)$$

*where  $(\bar{F} \mathbf{\Pi})(x) = \int_{\mathbb{R}_+} \bar{F}(u) \Pi(u, x) du$  and  $(\bar{G} \mathbf{\Pi})(x) = \int_{\mathbb{R}_+} \bar{G}(u) \Pi(u, x) du$ . The equality occurs iff  $\bar{F} \mathbf{\Pi} = \bar{F}$  and  $\bar{G} \mathbf{\Pi} = \bar{G}$ .*

**Theorem 1.6** *Let  $\mathbf{X}_1^3$  be a triple of RVs with joint SF  $\bar{F}(\mathbf{x}_1^3)$ . Let  $\mathbf{x}_1^3 = (x_1, x_2, x_3) \in \mathbb{R}_+^3 \mapsto \phi(\mathbf{x}_1^3)$  be a WF such that  $X_1$  and  $X_3$  are conditionally independent given  $X_2$  modulo  $\phi$ .*

(a) (Data-processing inequality for conditional WCRE). *Assume inequality (1.21) by swapping  $X_2$  with  $X_1$ . Then the following assertion for conditional WCREs holds true:*

$$\mathcal{E}_{\psi_{23}}^w(X_3|X_2) \leq \mathcal{E}_{\psi_{13}}^w(X_3|X_1), \quad (1.25)$$

*with equality iff  $X_2$  and  $X_3$  are independent modulo  $\phi$ . In addition assume that given WF  $\phi$  and triple of RVs  $\mathbf{X}_1^3$  obey*

$$\int_{\mathbb{R}_+^3} \phi(\mathbf{x}_1^3) \bar{F}(\mathbf{x}_1^3) \left[ \bar{F}_{2|13}(x_2|\mathbf{x}_{13} - 1) \right] d\mathbf{x}_1^3 \leq 0 \quad (1.26)$$

*Then*

$$\mathcal{E}_{\psi_{13}}^w(X_3|X_1) \leq 2\mathcal{E}_{\psi_{23}}^w(X_3|X_2); \quad (1.27)$$

(b) (Data-processing inequality for mutual WCRE). Assume inequality (1.28):

$$\int_{\mathbb{R}_+^3} \phi(\mathbf{x}_1^3) \left[ \bar{F}(\mathbf{x}_1^3) - \bar{F}_3(x_3) \prod_{i=1,2} \bar{F}_{i|3}(x_i|x_3) \right] d\mathbf{x}_1^3 \geq 0 \quad (1.28)$$

Then

$$\tau_{\psi_{13}}^w(X_1 : X_3) \leq \tau_{\psi_{12}}^w(X_1 : X_2). \quad (1.29)$$

Here, equality in (1.29) holds iff, modulo  $\phi$ , RVs  $X_1$  and  $X_2$  are conditionally independent given  $X_3$ .

**Theorem 1.7** (Concavity of the WCRE). Given WF  $\phi$ , set  $\phi'(x) = \frac{d}{dx}\phi(x)$  and  $\phi''(x) = \frac{d^2}{dx^2}\phi(x)$ . The functional  $\bar{F} \mapsto \mathcal{E}_\phi^w(1 - \bar{F})$  is concave function in  $\bar{F}$  under following suppositions:

- (i) The WF  $\phi$  is non-increasing (non-decreasing) for  $x \in [e^{-1}, 1]$  ( $x \in [0, e^{-1}]$ ).
- (ii) For  $x \in [0, 1]$

$$\phi''(\bar{F}^{-1}(x)) - \frac{f'(\bar{F}^{-1}(x))}{f(\bar{F}^{-1}(x))} \phi'(\bar{F}^{-1}(x)) \leq 0, \quad (1.30)$$

here  $f'$  denotes the derivative of  $f$  w.r.t.  $x$ .

**Proof.** Set  $g(x) = x \log x$ ,  $x \in [0, 1]$ . To implement the concavity property for WCRE, it is sufficient to prove the function  $\phi(\bar{F}^{-1}(x)).g(x)$ ,  $x \in [0, 1]$  is convex. Therefore we compute

$$\frac{d}{dx} \phi(\bar{F}^{-1}(x)).g(x) = \left( \frac{d}{dx} \phi(\bar{F}^{-1}(x)).g(x) + \phi(\bar{F}^{-1}(x)).\frac{d}{dx} g(x) \right). \quad (1.31)$$

and

$$\begin{aligned} \frac{d^2}{dx^2} \phi(\bar{F}^{-1}(x)).g(x) \\ = \left( \frac{d^2}{dx^2} \phi(\bar{F}^{-1}(x)) \right).g(x) + \left( \frac{d^2}{dx^2} g(x) \right). \phi(\bar{F}^{-1}(x)) + 2 \frac{d}{dx} g(x). \frac{d}{dx} \phi(\bar{F}^{-1}(x)). \end{aligned} \quad (1.32)$$

Evidently the middle expression in RHS of above inequality is non-negative. Furthermore, note that

$$\begin{aligned} \frac{d}{dx} \phi(\bar{F}^{-1}(x)) &= -\frac{\phi'(\bar{F}^{-1}(x))}{f(\bar{F}^{-1}(x))}, \\ \frac{d^2}{dx^2} \phi(\bar{F}^{-1}(x)) &= \frac{1}{f^2(\bar{F}^{-1}(x))} \left[ \phi''(\bar{F}^{-1}(x)) - \frac{f'(\bar{F}^{-1}(x))}{f(\bar{F}^{-1}(x))} \phi'(\bar{F}^{-1}(x)) \right]. \end{aligned} \quad (1.33)$$

Combining (1.32) and (1.33), under assumptions (i) and (ii) we conclude the result.  $\square$



## 2 Additional results

Following steps in the proof of Theorem 1 from [7], we propose the following theorem.

**Theorem 2.1** *Assume for given  $0 < a < \infty$  the following integrals are finite:*

$$\int_{(0,a)^n} \phi(\mathbf{x}) d\mathbf{x} < \infty \quad \text{and} \quad \int_{\mathbb{R}_+^n / (0,a)^n} \phi(\mathbf{x}) \prod_{i=1}^n x_i^{-\frac{p\alpha}{n}} d\mathbf{x} < \infty \quad (2.1)$$

Then  $\mathcal{E}_\phi^w(\mathbf{X}) < \infty$  if for all  $i, p$  and some  $0 \leq \alpha \leq 1$ ,  $\mathbb{E}[X_i^p] < \infty$ .

Furthermore, set  $\psi(x) = \int_0^x \phi(t) dt$ , in particular,  $\phi(\mathbf{x}) = \prod_{i=1}^n \phi(x_i)$ . Then for all  $a > 0$  the assumptions (2.1) take the form:

$$\int_a^\infty \phi(x_i) x_i^{-\frac{p\alpha}{n}} dx_i < \infty, \quad \psi(a) - \psi(0) < \infty.$$

**Proof.** Following arguments given in [7], we using Hölder's inequality. Recall Step 2 in the Theorem 1 of [7]. For  $0 \leq \alpha \leq 1$  we have

$$P[X_i > x_i, 1 \leq i \leq n] |\log P[X_i > x_i, 1 \leq i \leq n]| \leq \frac{e^{-1}}{1-\alpha} \prod_{i=1}^n P[X_i > x_i]^{\frac{\alpha}{n}}.$$

By multiplying both sides of above inequality in  $\phi(\mathbf{x})$  and then integrating on  $\mathbb{R}_+^n$ , we obtain

$$\mathcal{E}_\phi^w(\mathbf{X}) \leq \frac{e^{-1}}{1-\alpha} \int_{\mathbb{R}_+^n} \phi(\mathbf{x}) \prod_{i=1}^n P[X_i > x_i]^{\frac{\alpha}{n}} d\mathbf{x}.$$

Furthermore,

$$\mathcal{E}_\phi^w(\mathbf{X}) \leq \frac{e^{-1}}{1-\alpha} \left[ \int_{(0,a)^n} \phi(\mathbf{x}) d\mathbf{x} + \int_{\mathbb{R}_+^n / (0,a)^n} \phi(\mathbf{x}) \prod_{i=1}^n P[X_i > x_i]^{\frac{\alpha}{n}} d\mathbf{x} \right]. \quad (2.2)$$

Owing to Markov inequality for  $p \geq 0$ , the last term of RHS in (2.2) is less and equal than:

$$\frac{e^{-1}}{1-\alpha} \left[ \prod_{i=1}^n \mathbb{E}[X_i^p]^{\frac{\alpha}{n}} \right] \int_{\mathbb{R}_+^n / (0,a)^n} \phi(\mathbf{x}) \prod_{i=1}^n x_i^{-\frac{p\alpha}{n}} d\mathbf{x}.$$

By virtue of (2.1) this leads directly to the result.  $\square$

**Remark.** Note that in case  $\phi(\mathbf{x}) = \prod_{i=1}^n \phi(x_i)$ , the (2.2) reads

$$\begin{aligned} \mathcal{E}_\phi^w(X) &\leq \frac{e^{-1}}{1-\alpha} \prod_{i=1}^n \int_0^\infty \phi(x_i) P[X_i > x_i]^{\frac{\alpha}{n}} dx_i \\ &\leq \frac{e^{-1}}{1-\alpha} \prod_{i=1}^n \left\{ (\psi(a) - \psi(0)) + \mathbb{E}[X_i^p]^{\frac{\alpha}{n}} \int_a^\infty \phi(x_i) x_i^{-\frac{p\alpha}{n}} dx_i \right\}. \end{aligned}$$

Using the method in Theorem 5, [8] and arguments in Theorem 2.1, the following result is given. The proof of Theorem 2.2 is similar to Theorem 5 in [8] and omitted.

**Theorem 2.2** (Cf. Theorem 5 from [8].) *Let the random vector  $\mathbf{X}_k$  converges in distribution to the random vector  $\mathbf{X}$ . Also suppose that  $\phi$  is a WF whereas (2.1) holds true. If all  $\mathbf{X}_k$  are bounded in  $L^p$  then*

$$\lim_k \mathcal{E}_\phi^w(\mathbf{X}_k) = \mathcal{E}_\phi^w(\mathbf{X}). \quad (2.3)$$

Now we focus on the sum of independent RVs: The standard Shannon and cumulative entropies of a sum of independent RVs is larger than and equal of each . We show as analogues as Theorem 2 in [7], the same result is fulfilled for WCRE either.

**Theorem 2.3** *Consider two non-negative and independent RVs  $X$  and  $Y$  with PDFs  $f_X$  and  $f_Y$ , respectively. Then*

$$\max \left\{ \mathcal{E}_{\psi_Y}^w(X), \mathcal{E}_{\psi_X}^w(Y) \right\} \leq \mathcal{E}_\phi^w(X + Y).$$

Here  $\psi_Y(x) = \int f_Y(y)\phi(x+y)dy$  and swap  $X$  with  $Y$  in  $\psi_X$ .

**Proof.** We again follow the argument from [7]. By using Jensen's inequality, write:

$$\begin{aligned} & P[X + Y > w] \log P[X + Y > w] \\ & \leq \int f_Y(y) P[X > w - y] \log P[X > w - y] dy. \end{aligned} \quad (2.4)$$

Multiply both sides by  $\phi(w)$  and then integrate with respect to  $w$  from 0 to  $\infty$ :

$$\begin{aligned} -\mathcal{E}_\phi^w(X + Y) & \leq \int f_Y(y) \int_0^\infty \phi(w) P[X > w - y] \log P[X > w - y] dw dy \\ & = \int f_Y(y) \int_y^\infty \phi(w) P[X > w - y] \log P[X > w - y] dw dy \\ & = \int f_Y(y) \int_0^\infty \phi(w + y) P[X > w] \log P[X > w] dw dy \\ & = \int_0^\infty \left[ \int \phi(w + y) f_Y(y) dy \right] P[X > w] \log P[X > w] dw = -\mathcal{E}_{\psi_Y}^w(X). \end{aligned}$$

The first equality here is obtained because  $X$  is a non-negative RV. Consequently, for  $w < y$ ,  $P[X > w - y] = 1$ .  $\square$

In addition, the following extended assertion of Theorem 4 from [7] holds true (and is straightforward).

**Theorem 2.4** For given independent RV  $X_i$ , we have

$$\mathcal{E}_\phi^w(\mathbf{X}) = \sum_i \mathcal{E}_{\phi_i^*}^w(X_i), \quad (2.5)$$

where  $\phi_i^* = \int \phi(\mathbf{x}) \prod_{j \neq i} \bar{F}(x_j) d\mathbf{x}_1^{i-1} d\mathbf{x}_{i+1}^n$ . In a particular case  $\phi(\mathbf{x}) = \prod_{j=1}^n \phi_j(x_j)$ , set  $\psi_j(x_j) = \int_0^{x_j} \phi_j(t) dt$ . Then  $\mathcal{E}_\phi^w(\mathbf{X})$  in (1.7) becomes:

$$\mathcal{E}_\phi^w(\mathbf{X}) = \sum_{i=1}^n \left( \prod_{j=1, j \neq i}^n \left[ \mathbb{E}_{X_j}(\psi_j(X_j)) - \psi_j(0) \right] \right) \mathcal{E}_{\phi_i}^w(X_i).$$

### 3 Bounds for the WCRE

In this section, our goal is to establish additional bounds for the WCRE. First, let us show how the WCRE can be dominated by the standard entropy, as well as by the CRE; cf. [7].

**Theorem 3.1** Let  $h(X)$  be the Shannon entropy of a non-negative RV  $X$  having PDF  $f$  and SF  $\bar{F}$ . Then

$$\mathcal{E}_\phi^w(X) \geq \alpha_\phi \exp\{h(X)\}, \quad (3.1)$$

where

$$\alpha_\phi = \exp \left\{ \mathbb{E}[\log \phi(X)] + \int_0^1 \log(x|\log x|) dx \right\}. \quad (3.2)$$

**Proof.** The proof follows directly from log-sum inequality:

$$\begin{aligned} & \int f(x) \log \frac{f(x)}{\phi(x)\bar{F}(x)|\log \bar{F}(x)|} dx \\ & \geq \left( \int f(x) dx \right) \log \left( \frac{\int f(x) dx}{\int \phi(x)\bar{F}(x)|\log \bar{F}(x)| dx} \right) = \log \frac{1}{\mathcal{E}_\phi^w(X)}. \end{aligned}$$

Note that if  $\int \phi(x)\bar{F}(x) \log \bar{F}(x) dx = \infty$ , the proof is trivial. The LHS leads:

$$-h(X) - \mathbb{E}[\log \phi(X)] - \int_0^1 \log x |\log x| dx.$$

Consequently,

$$\log \mathcal{E}_\phi^w(X) \geq h(X) + \mathbb{E}[\log \phi(X)] + \int_0^1 \log x |\log x| dx.$$

This completes the proof.  $\square$

Let  $(X, Y)$  be a pair of RVs with a conditional SF  $\bar{F}(x|y)$ . Moreover assume an additional WF  $(x, y) \in \mathbb{R}_+^2 \mapsto \varphi(x, y) \geq 0$ . The WCRE of RV  $X$  given  $Y = y$  with WF  $\varphi(x, y)$  is defined by

$$\mathcal{E}_\varphi^w(X|Y = y) = - \int \varphi(x, y) \bar{F}(x|y) \log \bar{F}(x|y) \, dx. \quad (3.3)$$

Later, owing to (3.3), the generalized statement of Theorem 3.1 with similar proof is driven, therefore we omit the proof.

**Lemma 3.1** *For a non-negative RV  $X$ , let  $f(x|y)$  be the conditional PDF  $X$  given  $Y = y$ . Set*

$$\alpha_\varphi(y) = \exp \left\{ \mathbb{E}_{X|Y=y}[\log \varphi(X, y)] + \int_0^1 \log(x|\log x|) \, dx \right\}$$

*Then*

$$\mathcal{E}_\varphi^w(X|Y = y) \geq \alpha_\varphi(y) \exp\{h(X|Y = y)\}. \quad (3.4)$$

*where  $h(X|Y = y)$  is Shannon entropy of  $X$  given  $Y = y$ .*

**Definition 3.1** *Given WF  $(x, y) \in \mathbb{R}_+^2 \mapsto \varphi(x, y) \geq 0$ , set  $\bar{\phi}(x) = \int_0^\infty \varphi(x, y) f(y|x) \, dy$ . The Cross WCRE is introduced by*

$$\mathcal{E}_\varphi^w(X, Y) = \mathcal{E}_\phi^w(X) + \mathbb{E}_X[\mathcal{E}_\varphi^w(Y|X = x)]. \quad (3.5)$$

**Lemma 3.2** *Assume WF  $(x, y) \in \mathbb{R}_+^2 \mapsto \varphi(x, y) \geq 1$  and set*

$$\alpha_\varphi^* = \exp \left\{ \mathbb{E}_{X,Y}[\log \phi(X, Y)] + \int_0^1 \log(x|\log x|) \, dx \right\}. \quad (3.6)$$

*Then*

$$\mathcal{E}_\phi^w(X, Y) \geq 2\alpha_\varphi^* \exp\left\{\frac{h(X, Y)}{2}\right\}.$$

**Proof.** Following the assumption  $\varphi \geq 1$ , we observe

$$\mathbb{E}_X[\log \bar{\phi}(X)] \geq \mathbb{E}_{X,Y}[\log \varphi(X, Y)]. \quad (3.7)$$

Owing to the convexity of  $e^x$  and Jensen inequality, we have

$$\mathbb{E}[\mathcal{E}_\varphi^w(Y|X = x)] \geq \alpha_\varphi^* \exp\{h(Y|X)\}. \quad (3.8)$$

where  $h(Y|X)$  is denoted for conditional standard entropy.

$$\begin{aligned}
\mathcal{E}_\varphi^w(X, Y) &= \mathcal{E}_\phi^w(X) + \mathbb{E}_X [\mathcal{E}_\varphi^w(Y|X = x)] \\
&\geq \alpha_{\bar{\phi}} \exp\{h(X)\} + \alpha_\varphi^* \exp\{h(Y|X)\} \\
&\geq 2\alpha_\varphi^* \exp\{\frac{h(X, Y)}{2}\}.
\end{aligned} \tag{3.9}$$

Here  $\alpha_{\bar{\phi}}$  is defined as (3.2) by replacing  $\bar{\phi}$  in  $\phi$ . The first inequality in (3.9) drives from (3.1) and (3.8). The second inequality holds by using (3.7) and  $2 \exp(\frac{t+s}{2}) \leq \exp(t) + \exp(s)$ .  $\square$

**Lemma 3.3** (Cf. Proposition 4 from [7].) *Let  $X$  be a non-negative continuous RV. Given WF  $\phi$ , suppose that  $\psi(x) = \int_0^x \phi(t)dt$  is bounded. There exist a function  $Y = g(X)$  such that:*

(i) *The WCRE and the weighted entropy (WE) are related by*

$$h_\phi^w(Y) = \frac{\mathcal{E}_\phi^w(X)}{\mathbb{E}(X)} + \frac{\mathbb{E}(\psi(X)) - \psi(0)}{\mathbb{E}(X)} \log \mathbb{E}(X),$$

(ii) *Assume  $\psi(0) = 0$ , then the WCRE and the Shannon entropy (SE) are related by*

$$h(Y) = \frac{\mathcal{E}_\phi^w(X)}{\mathbb{E}(\psi(X))} - \frac{\Theta}{\mathbb{E}(\psi(X))} + \log \mathbb{E}(\psi(X)),$$

$$\text{here } \Theta = \int_0^\infty \phi(x) \log \phi(x) \bar{F}(x) dx.$$

**Proof.** The proof is straightforward by considering the CDF,  $F$ , as an RV having PDF  $\frac{P(X > x)}{\mathbb{E}(X)}$  and  $\frac{\phi(x)P(X > x)}{\mathbb{E}(\psi(X))}$ , respectively. Next use the definitions of SE and WE.

Note that simply by choosing  $g(x) = F^{-1}(F(x))$  we can find a  $g$ .  $\square$

Next we present a Lower bound for WCR, the origin of this Lemma goes back to Proposition 1 from [8].

**Lemma 3.4** *Let  $X$  and  $Y$  be two iid RVs. Also For given WF  $\phi$  set  $\psi(x) = \int_0^x \phi(t)dt$ . We obtain*

$$2\mathcal{E}_\phi^w(X) \geq \mathbb{E}[|\psi(X) - \psi(Y)|]. \tag{3.10}$$

*In particular, suppose that  $X$  is a non-negative RV, then*

$$2\mathcal{E}_\phi^w(X) \geq \mathbb{E}[|\psi(X) - \mathbb{E}[\psi(X)]|]. \tag{3.11}$$

**Proof.** According to Proposition 1, [8], similarly we derive:

$$2\bar{F}(x) - 2\bar{F}^2(x) = \mathbb{P}[\max\{X, Y\} > x] - \mathbb{P}[\min\{X, Y\} > x], \quad (3.12)$$

multiplying both sides of (3.12) in  $\phi(x)$  and then integrating from zero to infinity:

$$\begin{aligned} & 2 \int_0^\infty \phi(x) \bar{F}(x) (1 - \bar{F}(x)) dx \\ &= \int_0^\infty \phi(x) \mathbb{P}[\max\{X, Y\} > x] - \int_0^\infty \phi(x) \mathbb{P}[\min\{X, Y\} > x]. \end{aligned}$$

Next using integrate by part in RHS , we can write

$$2 \int_0^\infty \bar{F}(x) |\log \bar{F}(x)| dx \geq \mathbb{E}[|\psi(X) - \psi(Y)|]. \quad (3.13)$$

The LHS can be modified because of  $x(1-x) \leq x|\log x|$ . The inequality (3.13) proves (3.10). Moreover the assertion (3.11) follows directly from:

$$\mathbb{E}[|\psi(X) - \psi(Y)|] \geq \mathbb{E}[|\psi(X) - \mathbb{E}[\psi(X)]|]. \quad \square$$

**Lemma 3.5** (Cf. Proposition 2 from [8].) *Let  $X$  be a non-negative RV. Then for function  $\psi$  defined as in Lemma 3.4:*

$$\mathcal{E}_\phi^w(X) = \mathbb{E}\left[(\psi(0) - \psi(X))(1 + \log \bar{F}(X))\right]. \quad (3.14)$$

**Proof.** The proof is straightforward and based on the equality:

$$\bar{F}(x) \log \bar{F}(x) = - \int_x^\infty (1 + \log \bar{F}(t)) d\bar{F}(t). \quad \square$$

**Remark:** More application of conjugate or the Fenchel Transform of the convex function  $x \log x$  is  $\exp(y - 1)$ , that is

$$\exp(y - 1) = \sup [xy - x \log x : 0 < x < \infty].$$

Consequently, for non-negative RVs  $X$  and  $Y$ :

$$xy \leq x \log x + \exp(y - 1).$$

If we use this inequality, emerging the definition WCRE, an upper bound for WCRE in terms of  $|\psi(X) - \mathbb{E}[\psi(X)]|$  is given:

$$\mathcal{E}_\phi^w(X) \leq 2\mathbb{E}\left[|\psi(X) - \mathbb{E}[\psi(X)]| \log |\psi(X) - \mathbb{E}[\psi(X)]|\right] + \frac{4}{e}.$$

Here  $\psi$  is defined as before.

**Theorem 3.2** (Cf. Theorem 1 from [8].) *Suppose that  $X$  is a non-negative RV. Set  $\psi(x) = \int_0^x \phi(t)dt$  and  $\psi^{-1}$  the inverse function of  $\psi$ . Then*

$$\begin{aligned} \mathbb{E}[\psi(X) \log^+ \psi(x)] \\ \leq \mathcal{E}_\phi^w(X) + \mathbb{E}[\psi(X) \mathbf{1}(X > \psi^{-1}(1))] \log \left( e \mathbb{E}[\psi(X) \mathbf{1}(X > \psi^{-1}(1))] \right). \end{aligned} \quad (3.15)$$

*This implies:  $\mathbb{E}[\psi(X) \log^+ \psi(x)] < \infty$  if WCRE is finite.*

**Proof.** Following standard calculations, (see [8]), we can write

$$\begin{aligned} \mathbb{E}[\psi(X) \log^+ \psi(x)] \\ = \mathbb{E}[\psi(X) \mathbf{1}(X > \psi^{-1}(1))] - \bar{F}(\psi^{-1}(1)) + \int_{\psi^{-1}(1)}^{\infty} \phi(x) \bar{F}(x) \log \psi(x) dx. \end{aligned} \quad (3.16)$$

Moreover, for  $t > \psi^{-1}(1)$  one yields:

$$\psi(t) \mathbb{P}(X > t) \leq \mathbb{E}[\psi(X) \mathbf{1}(X > t)] \leq \mathbb{E}[\psi(X) \mathbf{1}(X > \psi^{-1}(1))].$$

Therefore, we obtain

$$\begin{aligned} \int_{\psi^{-1}(1)}^{\infty} \phi(x) \bar{F}(x) \log \psi(x) dx \\ \leq \mathcal{E}_\phi^w(X) + \log \mathbb{E}[\psi(X) \mathbf{1}(X > \psi^{-1}(1))] \int_{\psi^{-1}(1)}^{\infty} \phi(x) \bar{F}(x) dx. \end{aligned}$$

Finally according to (3.16), we get

$$\mathbb{E}[\psi(X) \log^+ \psi(x)] \leq \mathcal{E}_\phi^w(X) + \varsigma \left( 1 + \log \mathbb{E}[\psi(X) \mathbf{1}(X > \psi^{-1}(1))] \right).$$

Here  $\varsigma = \mathbb{E}[\psi(X) \mathbf{1}(X > \psi^{-1}(1))] - \bar{F}(\psi^{-1}(1))$ . The inequality (3.15) holds true then.  $\square$

## 4 Maximum WCRE properties

**Theorem 4.1** *Suppose  $x \in \mathbb{R}_+ \mapsto \phi(x) \geq 0$  is given WF. Then  $\bar{F}^m$  maximizes the WCRE  $\mathcal{E}_\phi^w(\bar{F})$ , modulo  $\phi$ , uniquely when the following constraints are fulfilled:*

$$\int_{\mathbb{R}_+} \phi(x) [\bar{F}(x) - \bar{F}^m(x)] dx \geq 0 \quad \text{and} \quad \int_{\mathbb{R}_+} \phi(x) [\bar{F}(x) - \bar{F}^m(x)] \log \bar{F}^m(x) dx \geq 0. \quad (4.1)$$

**Example 4.1** Consider a random vector  $\mathbf{X}_1^n = (X_1, \dots, X_n) : \Omega \rightarrow \mathbb{R}^n$  with PDF  $f$ , the PDF  $F$  and the SF  $\bar{F}$ , mean vector  $\mu = (\mu_1, \dots, \mu_n)$  with  $\mathbb{E}X_i = \mu_i$  and covariance matrix  $\mathbf{C} = (C_{ij})$  with  $C_{ij} = \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)]$ ,  $1 \leq i, j \leq n$ . Let  $f^{\text{No}}$  be the normal PDF with the same  $\mu$  and  $\mathbf{C}$  and  $\bar{F}^{\text{No}}$  be the normal SF. Introduce

$$\alpha^*(\mathbf{x}) = \int_{\mathbf{x}}^{\infty} \exp \left\{ -\frac{1}{2}(\mathbf{t} - \mu)^T \mathbf{C}^{-1}(\mathbf{t} - \mu) \right\} d\mathbf{t}. \quad (4.2)$$

Then

$$\rho(\mathbf{C}) := \bar{F}^{\text{No}}(\mathbf{x}) = (2\pi)^{-n/2} (\det \mathbf{C})^{-1/2} \alpha^*(\mathbf{x}). \quad (4.3)$$

Given a WF  $\mathbf{x}_1^n = (x_1, \dots, x_n) \in \mathbb{R}^n \mapsto \phi(\mathbf{x}_1^n) \geq 0$ , suppose that

$$\begin{aligned} \int_{\mathbb{R}_+^n} \phi(\mathbf{x}) [\bar{F}(\mathbf{x}) - \bar{F}^{\text{No}}(\mathbf{x})] d\mathbf{x} &\geq 0 \quad \text{and} \\ \log [(2\pi)^{n/2} (\det \mathbf{C})^{1/2}] \int_{\mathbb{R}_+^n} \phi(\mathbf{x}) [\bar{F}(\mathbf{x}) - \bar{F}^{\text{No}}(\mathbf{x})] d\mathbf{x} &- \int_{\mathbb{R}_+^n} \phi(\mathbf{x}) [\bar{F}(\mathbf{x}) - \bar{F}^{\text{No}}(\mathbf{x})] \log \alpha^*(\mathbf{x}) d\mathbf{x} \leq 0. \end{aligned} \quad (4.4)$$

Then

$$\begin{aligned} \mathcal{E}_{\phi}^w(F) &\leq \mathcal{E}_{\phi}^w(F^{\text{No}}) \\ &= \frac{1}{2} \log [(2\pi)^n (\det \mathbf{C})] \int_{\mathbb{R}_+^n} \phi(\mathbf{x}) \bar{F}^{\text{No}}(\mathbf{x}) d\mathbf{x} - \int_{\mathbb{R}_+^n} \phi(\mathbf{x}) \bar{F}^{\text{No}}(\mathbf{x}) \log \alpha^*(\mathbf{x}) d\mathbf{x}. \end{aligned} \quad (4.5)$$

with equality iff  $\bar{F} = \bar{F}^{\text{No}}$  modulo  $\phi$ .

**Example 4.2** Let  $F^{\text{Exp}}$  and  $\bar{F}^{\text{Exp}}$  be respectively CDF and SF on  $\mathbb{R}_+$  with mean  $\frac{1}{\lambda}$ . Suppose the following constraints are fulfilled:

$$\int_{\mathbb{R}_+} \phi(x) [\bar{F}(x) - \bar{F}^{\text{Exp}}(x)] dx \geq 0 \quad \text{and} \quad \int_{\mathbb{R}_+} x \phi(x) [\bar{F}(x) - \bar{F}^{\text{Exp}}(x)] dx \leq 0, \quad (4.6)$$

where  $x \in \mathbb{R}_+ \mapsto \phi(x) \geq 0$  is a given WF. Then

$$\mathcal{E}_{\phi}^w(F) \leq \mathcal{E}_{\phi}^w(F^{\text{Exp}}) = \lambda \int_{\mathbb{R}_+} x \phi(x) e^{-\lambda x} dx. \quad (4.7)$$

and  $\bar{F}^{\text{Exp}}$  is a unique maximizer modulo  $\phi$ .

The next Theorem is a direct result of Theorem 1.7 and Example 4.1.



**Theorem 4.2** (The weighted Ky Fan inequality using the WCRE; cf. [11], Theorem 3.2). *Assume for given  $\lambda_1, \lambda_2 \in [0, 1]$  with  $\lambda_1 + \lambda_2 = 1$  and positive definite matrices  $\mathbf{C}_1, \mathbf{C}_2$  and  $\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2$  the assumption in Theorem 1.7, (i) and (ii) hold true. Furthermore*

$$\begin{aligned} & \int_{\mathbb{R}_+^n} \phi(\mathbf{x}) [\lambda_1 \bar{F}_{\mathbf{C}_1}^0(\mathbf{x}) + \lambda_2 \bar{F}_{\mathbf{C}_2}^0(\mathbf{x}) - \bar{F}_{\mathbf{C}}^0(\mathbf{x})] d\mathbf{x} \geq 0 \quad \text{and} \\ & \log [(2\pi)^{n/2} (\det \mathbf{C})^{1/2}] \int_{\mathbb{R}_+^n} \phi(\mathbf{x}) [\lambda_1 \bar{F}_{\mathbf{C}_1}^0(\mathbf{x}) + \lambda_2 \bar{F}_{\mathbf{C}_2}^0(\mathbf{x}) - \bar{F}_{\mathbf{C}}^0(\mathbf{x})] d\mathbf{x} \\ & - \int_{\mathbb{R}_+^n} \phi(\mathbf{x}) [\lambda_1 \bar{F}_{\mathbf{C}_1}^0(\mathbf{x}) + \lambda_2 \bar{F}_{\mathbf{C}_2}^0(\mathbf{x}) - \bar{F}_{\mathbf{C}}^0(\mathbf{x})] \log \alpha_{\mathbf{C}}^*(\mathbf{x}) d\mathbf{x} \leq 0. \end{aligned} \quad (4.8)$$

are fulfilled. Then

$$\rho(\lambda_1 \mathbf{C}_1 + \lambda_2 \mathbf{C}_2) - \lambda_1 \rho(\mathbf{C}_1) - \lambda_2 \rho(\mathbf{C}_2) \geq 0. \quad (4.9)$$

with equality iff  $\lambda_1 \lambda_2 = 0$  or  $\mathbf{C}_1 = \mathbf{C}_2$ .

**Lemma 4.1** Let  $\mathbf{X}_1^n = (X_1, \dots, X_n)$  be a random vector, with components  $X_i : \Omega \rightarrow \mathcal{X}_i$ ,  $1 \leq i \leq n$ , and the joint SF  $\bar{F}$ . Introduce the random vector  $\bar{\mathbf{x}}_i = (\mathbf{x}_1^{i-1}, \mathbf{x}_{i+1}^n)$ ,  $\bar{F}_i(x_i)$  the marginal SF for RV  $X_i$ :

$$\bar{F}_i(x_i) = \lim_{\bar{\mathbf{x}}_i \rightarrow \infty} \bar{F}(\mathbf{x}) \quad \text{and} \quad \bar{F}_{|i}(\mathbf{x}_1^n | x_i) = \frac{\bar{F}(\mathbf{x}_1^n)}{\bar{F}_i(x_i)}.$$

For given a WF  $\phi$ , suppose that

$$\int_{\mathbb{R}_+^n} \phi(\mathbf{x}) \left[ \bar{F}(\mathbf{x}) - \prod_{i=1}^n \bar{F}_i(x_i) \right] d\mathbf{x} \geq 0. \quad (4.10)$$

Then

$$\mathcal{E}_{\phi}^w(\mathbf{X}) \leq \sum_{i=1}^n \mathcal{E}_{\psi_i}^w(X_i). \quad (4.11)$$

where

$$\psi_i(x_i) = \int_{\mathbb{R}_+^{n-1}} \phi(\mathbf{x}_1^n) \bar{F}_{|i}(\mathbf{x}_1^n | x_i) d\mathbf{x}_1^{i-1} d\mathbf{x}_{i+1}^n. \quad (4.12)$$

The equality in (4.11) holds true holds iff, modulo  $\phi$ , components  $X_1, \dots, X_n$  are independent.

In the following theorem a straightforward application of Lemma 4.1 is given.

**Theorem 4.3** (The weighted Hadamard inequality using the WCRE; cf. [11], Theorem 3.3). *Let  $\mathbf{C} = (C_{ij})$  be a positive definite  $n \times n$  matrix and  $\bar{F}_{\mathbf{C}}^{\text{No}}$  the normal SF with the zero mean vector and the covariance matrix  $\mathbf{C}$ . For given WF  $\mathbf{x}_1^n = (x_1, \dots, x_n) \in \mathbb{R}^n \mapsto \phi(\mathbf{x}_1^n)$ , introduce  $\alpha^*(\mathbf{x})$  by (4.2) and*

$$\alpha_i^*(x) = \int_x^\infty e^{-t^2/2C_{ii}} dt \quad \text{and} \quad \alpha = \int_{\mathbb{R}_+^n} \phi(\mathbf{x}) \bar{F}^{\text{No}}(\mathbf{x}) d\mathbf{x}. \quad (4.13)$$

Suppose that

$$\int_{\mathbb{R}_+^n} \phi(\mathbf{x}) \left[ \bar{F}^{\text{No}}(\mathbf{x}) - \prod_{i=1}^n \bar{F}_i^{\text{No}}(x_i) \right] d\mathbf{x} \geq 0. \quad (4.14)$$

Then

$$\frac{\alpha}{2} \log \left[ \prod_i C_{ii} / \det \mathbf{C} \right] + \int_{\mathbb{R}_+^n} \phi(\mathbf{x}) \bar{F}^{\text{No}}(\mathbf{x}) \log \left[ \alpha^*(\mathbf{x}) / \prod_i \alpha_i^*(x_i) \right] d\mathbf{x} \geq 0, \quad (4.15)$$

with equality iff  $\mathbf{C}$  is diagonal.

Next, we provide a characterization of the Weibull distribution using the maximum WCRE.

**Theorem 4.4** (Cf. Theorem 2 from [8].) *Suppose  $\psi_p^*(x) = \int_0^x t^p \phi(t) dt$  is a non-negative WF such that  $\psi(x) = \int_0^x \phi(t) dt$  and  $x \in \mathbb{R}^+ \mapsto \phi(x) \in [0, 1]$ . Among all non-negative RVs with given  $\mathbb{E}[\psi(X)]$  and  $\mathbb{E}[\psi_p^*(X)]$  the Weibull distribution  $W$  with SF  $\bar{F}_{\text{Wib}}(t) = \exp(-\lambda^q t^q)$ , has the maximal WCRE.*

Here the parameters  $q = p$  and

$$\lambda^q = \left( \frac{c_p}{\mathbb{E}[\psi(X)] - \psi(0)} \right)^p \cdot \left( \frac{\mathbb{E}[\psi_p^*(X)] - \psi_p^*(0)}{\mathbb{E}_{\text{Wib}}[\psi_q^*(X)] - \psi_q^*(0)} \right); \quad (4.16)$$

where  $c_p = \Gamma(1 + \frac{1}{p})$ .

**Proof.** The subsequence argument works by using Log-sum inequality once more. According to (21) in [8] but replacing  $\bar{G}(x) = \bar{F}_{\text{Wib}}(x) = \exp(-\mu^p x^p)$  in (1.5), we get

$$\mathcal{E}_\phi^w(X) \leq \left[ \mathbb{E}[\psi(X)] - \psi(0) \right] \log \frac{\mathbb{E}[\psi(X)] - \psi(0)}{\mu^{-1} c_p} + \mu^p \int_0^\infty \phi(t) t^p \bar{F}(t) dt.$$

Now choose  $\mu^{-1} c_p = \mathbb{E}[\psi(X)] - \psi(0)$ :

$$\mathcal{E}_\phi^w(X) \leq \frac{c_p^p (\mathbb{E}[\psi_p^*(X)] - \psi_p^*(0))}{\mathbb{E}[\psi(X)] - \psi(0)}.$$

Finally let  $q = p$  and  $\lambda$  as in (4.16), therefore we have

$$\mathcal{E}_\phi^w(X) \leq \lambda^q \left( \mathbb{E}_{\text{Wib}}[\psi_q^*(X)] - \psi_q^*(0) \right) = \mathcal{E}_\phi^w(Wib).$$

This completes the proof.  $\square$

**Theorem 4.5** Suppose that functions  $\psi$  and  $\phi$  are as in Theorem 3.2:  $\psi(x) = \int_0^x \phi(t)dt$ , and  $0 \leq \phi(x) \leq 1$ . Let  $X$  be a given non-negative RV. In addition assume  $Z := X(\lambda)$  is an exponentially distributed RV with mean  $\lambda^{-1} = \mathbb{E}[\psi(X)] - \psi(0)$ . If the constraints

$$\int_{\mathbb{R}_+} x \phi(x) [\bar{F}(x) - \bar{F}^{\text{Exp}}(x)] dx \geq 0. \quad (4.17)$$

holds true, then

$$\mathcal{E}_\phi^w(X) \leq \mathcal{E}_\phi^w(X(\lambda)) = \lambda \int_{\mathbb{R}_+} x \phi(x) \bar{F}^{\text{Exp}}(x) dx. \quad (4.18)$$

**Proof.** Using log-sum inequality we obtain

$$\begin{aligned} & \int_{\mathbb{R}_+} \phi(x) \bar{F}(x) \log(\phi(x) \bar{F}(x) e^{\lambda x}) dx \\ & \geq \left( \int_{\mathbb{R}_+} \phi(x) \bar{F}(x) dx \right) \log \left( \lambda \int_{\mathbb{R}_+} \phi(x) \bar{F}(x) dx \right). \end{aligned} \quad (4.19)$$

We also can write

$$\mathbb{E}[\psi(X)] - \psi(0) = \int_{\mathbb{R}_+} \phi(x) \bar{F}(x) dx,$$

therefore the expression (4.19) becomes

$$\begin{aligned} -\mathcal{E}_\phi^w(F) + \int_{\mathbb{R}_+} \bar{F}(x) \phi(x) \log \phi(x) dx + \lambda \int_{\mathbb{R}_+} x \phi(x) \bar{F}(x) dx \\ \geq (\mathbb{E}[\psi(X)] - \psi(0)) \left\{ \log \lambda + \log (\mathbb{E}[\psi(X)] - \psi(0)) \right\}. \end{aligned}$$

Equivalently

$$\begin{aligned} -\mathcal{E}_\phi^w(F) \geq & - \int_{\mathbb{R}_+} \bar{F}(x) \phi(x) \log \phi(x) dx - \lambda \int_{\mathbb{R}_+} x \phi(x) \bar{F}(x) dx \\ & + (\mathbb{E}[\psi(X)] - \psi(0)) \left\{ \log \lambda + \log (\mathbb{E}[\psi(X)] - \psi(0)) \right\}. \end{aligned} \quad (4.20)$$

Now, set

$$\varpi = \left( \mathbb{E}[\psi(X)] - \psi(0) \right)^2 \left( \int_{\mathbb{R}_+} x \phi(x) \bar{F}(x) dx \right)^{-1}, \quad \lambda^* = \frac{\varpi}{\mathbb{E}[\psi(X)] - \psi(0)}$$

It is admissible (4.20) is fulfilled for all positive  $\lambda$ , so is also valid for maximum value of  $\lambda = \lambda^*$ . This represents the formula:

$$\begin{aligned}
-\mathcal{E}_\phi^w(F) &\geq - \int_{\mathbb{R}_+} \bar{F}(x)\phi(x) \log \phi(x) dx - [\mathbb{E}[\psi(X)] - \psi(0)] + [\mathbb{E}[\psi(X)] - \psi(0)] \log \varpi \\
&\geq -[\mathbb{E}[\psi(X)] - \psi(0)] + [\mathbb{E}[\psi(X)] - \psi(0)] \log \varpi \\
&\geq -[\mathbb{E}[\psi(X)] - \psi(0)] + [\mathbb{E}[\psi(X)] - \psi(0)] \left\{ 1 - \frac{1}{\varpi} \right\} \\
&= -[\mathbb{E}[\psi(X)] - \psi(0)] \varpi^{-1}.
\end{aligned}$$

The second inequality holds true owing to  $\phi \in [0, 1]$  and the last inequality is satisfied by using  $\log x \geq 1 - \frac{1}{x}$ ,  $x \in \mathbb{R}_+$ .

Recalling assumption (4.17), leads to (4.18).  $\square$

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